Section 8.1 Relations and Function

Objective 1: Identify Independent and Dependent Variables

We have seen that applying mathematics to everyday life often involves situations in which one quantity is related to another. For example, the cost of fill a gas tank is related to the number of gallons purchased. The amount of simple interest paid on a loan is related to the amount owed. The total cost to manufacture 3D televisions is related to the number of televisions produced, and so on.

In Section 2.4, we solved formulas for a given variable. In doing so, we had to express a relationship between the given variable and any remaining variables. When we solve for a variable and it represents a unique value, that variable is called a dependent variable because its value depends on the value(s) of the remaining variable(s). Any remaining variables are called independent variables because we are free to select their values.

If the average price per gallon of regular unleaded gas is $3.24 on a given day, then two gallons would cost $3.24(2) = $6.38, three gallons would cost $3.24(3) = $9.72, and so on. We can model this situation with the equation

\[ y = 3.24x, \]

where \( y \) represents the cost in dollars and \( x \) represents the number of gallons purchased.

Since the equation is solved for \( y \), we identify \( y \) (cost) as the dependent variable and \( x \) (gallons of gas) as the independent variable. When an equation involving \( x \) and \( y \) is not solved for either variable, like the linear equation \( 2x + 3y = 12 \), we cannot specify which variable is independent or dependent. If we solve the equation for \( y \) (write in the slope-intercept form) as

\[ y = -\frac{2}{3}x + 4, \]

we then can say that \( y \) is the dependent variable and \( x \) is the independent variable. Or, if we solve the equation for \( x \) as

\[ x = -\frac{3}{2}y + 6, \]

we then can say that \( x \) is the dependent variable and \( y \) is the independent variable.

Example 1 Identifying Independent and Dependent Variables

For each of the following equations, identify the dependent variable and the independent variable(s).

a. \( y = 3x + 5 \)  
   b. \( w = ab + 3c^2 \)  
   c. \( 3x^2 + 9y = 12 \)
Solutions

a. Since the equation is solved for $y$, we identify $y$ as the dependent variable. The remaining variable, $x$, is the independent variable.

b. Since the equation is solved for $w$, we identify $w$ as the dependent variable. The remaining variables, $a$, $b$, and $c$, are independent variables.

c. Since the equation is not solved for either variable, we cannot identify either variable as dependent or independent.

In Example 1, we see that it is possible to have more than one independent variable in an equation. However, we will limit our discussion mainly to situations involving one dependent and one independent variable. Our gasoline cost model, $y = 3.24x$, is an example of such a situation.

Objective 2: Find the Domain and Range of a Relation

A relation is a relationship between two sets of numbers that can be represented by a set of ordered pairs.

Definition A relation is a set of ordered pairs.

In Section 3.2, we learned that equations in two variables define a set of ordered pair solutions. For example, if one gallon of regular unleaded gas is purchased, the total cost is $3.24(1) = $3.24, which gives an ordered pair $(1, 3.24)$. If two gallons are purchased, the total cost is $3.24(2) = $6.48, which gives the ordered pair $(2, 6.48)$. We can create more ordered pair solutions by following this process where we write the ordered pairs as (gallons, cost).

Since a graph is a visual representation of the ordered pair solutions to an equation, we consider equations and graphs to be relations because they define sets of ordered pairs.

At this point, you may wish to review set-builder notation and interval notation in Section 2.7.

Definitions
The domain of a relation is the set of all first coordinates.
The range of a relation is the set of all second coordinates.
Example 2  Finding the Domain and Range of a Relation

a. \{(-5, 7), (3, 5), (6, 7), (12, -4)\}  

b.  

\begin{align*}
(3, 4) \\
(2, 0) \\
(-2, -2) \\
(4, -5)
\end{align*}

Solutions  For parts a and b, identify the first coordinates of each ordered pair to find the domain and the second coordinates of each ordered pair to find the range.

a. Domain: \{-5, 3, 6, 12\};  Range: \{-4, 5, 7\}

b. Domain: \{-4, -2, 2, 3, 4\};  Range: \{-5, -2, 0, 3, 4\}

Example 3  Finding the Domain and Range of a Relation

Find the domain and range of each relation.

a. \(y = -x\)  

\begin{align*}
(-3, 4) \\
(-5, 2) \\
(5, 3) \\
(2, -3)
\end{align*}

b. \(y = x \wedge 2\)  

\begin{align*}
(3, 1)
\end{align*}

c. \(y = |x - 1|\)

Solutions  Try to find the domain and range of each relation on your own. Then check the solutions below.  \textbf{Interactive video p. 8.1-8}
a. The domain is the set of all first coordinate (or $x$) values for all points that lie on the graph, so the domain is $\{x \mid -5 \leq x \leq 5\}$ in set-builder notation or $[-5, 5]$ in interval notation. The range is the set of all second coordinate (or $y$) values for all points that lie on the graph, so the range is $\{y \mid -3 \leq y \leq 4\}$ in set-builder notation or $[-3, 4]$ in interval notation.

b. The arrows on the end of the graph show that the graph continues outward to the left both above and below the $x$-axis. The domain is the set of real numbers less than or equal to 3, represented as $\{x \mid x \leq 3\}$ or $(-\infty, 3]$. The range is the set of all real numbers, represented as $\mathbb{R}$ or $(-\infty, \infty)$.

c. To find the domain and range from an equation, it is helpful to first sketch the graph of the equation by plotting points. The domain is the set of all real numbers, $\mathbb{R}$ or $(-\infty, \infty)$. The range is the set of all real numbers, $y$, greater than or equal to zero, $\{y \mid y \geq 0\}$ or $[0, \infty)$.

When working with application problems, we often need to restrict the domain to use only those values that make sense within the context of the situation. This restricted domain is called the feasible domain. The feasible domain is the set of values for the first coordinates that make sense, or are feasible, in the context of the application. For example, in our gasoline cost model, $y = 3.24x$ with ordered pairs $(x, y)$, the domain of the equation is all real numbers. However, it does not make sense to use negative numbers in the domain since $x$ represents the number of gallons of gas purchased. Therefore, the feasible domain would be all real numbers greater than or equal to 0, written as $\{x \mid x \geq 0\}$ or $[0, \infty)$.

**Objective 3: Determine If Relations Are Functions**

A relation relates one set of numbers, the domain, to another, the range. When each value of the domain corresponds to (is paired with in an ordered pair) exactly one value in the range, we have a special type of relation called a function.

**Definition** A function is a special type of relation in which each value in the domain corresponds to exactly one value in the range.

Given a set of ordered pairs, we can determine if the relation is a function by looking at the first coordinates (usually the $x$-coordinate). If no first coordinate is repeated, then the relation is a function because each input value (first coordinate) corresponds to exactly one output value (second coordinate). If the same first coordinate corresponds to two or more different second coordinates, then the relation is not a function.

Given an equation, we can test input values to see if, when substitutes into the equation, there is more than one output value. In order for an equation to be a function, each input value must correspond to one and only one output value.
Example 4  Determining If Relations Are Functions

Determine if each of the following relations is a function. If it is not a function, give two ordered pairs from the relation that have the same first coordinate but different second coordinates. Assume the ordered pairs are in the form \((x, y)\), that is, \(x\) is the first coordinate.

a. \{(-3, 6), (2, 5), (0, 6), (17, -9)\}  
b. \{(4, 5), (7, -3), (4, 10), (-6, 1)\}

c. \{(-2, 3), (0, 3), (4, 3), (6, 3), (8, 3)\}  
d. \(|y - 5| = x + 3\)

e. \(y = x^2 - 3x + 2\)  
f. \(4x - 8y = 24\)

Solutions  First, try to determine which relations are functions on your own. Interactive video p. 8.1-10

a. Function  
b. Not a function  Four has two values 5 & 10. (4, 5) & (4, 10)

c. Function  
d. Not a function  One has two values 1 & 9. (1, 1) & (1, 9)

e. Function  
f. Function

Objective 4: Determine If Graphs Are Functions

If a relation appears as a graph, we can determine if the relation is a function by using the vertical line test. In mathematics, the horizontal axis (\(x\)-axis or axis for the first coordinate or independent variable) is usually used for the domain of a function and the vertical axis (\(y\)-axis or axis for the second coordinate or dependent variable) is usually used for the range of a function.

Vertical Line Test

If a vertical line intersects (crosses or touches) the graph of a relation at more than one point, then the relation is not a function. If every vertical line intersects the graph of a relation at no more than one point, then the relation is a function.

Why does the vertical line test work? Note that all the first coordinates are the same along a vertical line. So, two different points on a vertical line would have the same first coordinate but different second coordinates, hence, could not be points for a function. See illustration below. Or, Animation p. 8.1-11
In this diagram, the vertical line $x = -5$ intersects the graph of the relation in two points $(-5, 2)$ and $(-5, -2)$. Since the first coordinate $-5$ is paired with two different second coordinates $-2$ and $2$, the relation is *not* a function.

**Example 5 Determining If Graphs Are Functions**

Use the vertical line test to determine if each graph is a function.

<table>
<thead>
<tr>
<th></th>
<th>![Graph A]</th>
<th>![Graph B]</th>
<th>![Graph C]</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>Function</td>
<td>b. Not a function</td>
<td>c. Not a function</td>
</tr>
<tr>
<td>d.</td>
<td>Function</td>
<td>e. Function</td>
<td>f. Not a function</td>
</tr>
</tbody>
</table>

**Solutions** Apply the vertical line test to each graph on your own. *Animation p. 8.1-12*

a. Function  
b. Not a function  
c. Not a function  
d. Function  
e. Function  
f. Not a function

How many $x$-intercepts and $y$-intercepts can the graph of a function have?

The graph of a function can have at most one $y$-intercept. Remember that $y$-intercepts lie along the vertical $y$-axis. If a graph has more than one $y$-intercept, then it would be possible for a vertical line,
namely the line $x = 0$, to intersect the graph at more than one point at the same time. The graph would fail the vertical line test and would not be a function.

There is no limit to the number of $x$-intercepts the graph of a function can have.

It is possible for the graph of a function to have no intercepts, or only one type of intercept.

**Objective 5: Solve Application Problems Involving Relations and Functions**

Real-world situations can be described by relations and functions. We use mathematical models to describe these situations.

**Example 6  Video Entertainment and Sleep**

The data in the following table represent the average daily hours of sleep and average daily hours of video entertainment for six students at a local college.

<table>
<thead>
<tr>
<th>Video Entertainment</th>
<th>Sleep</th>
<th>Video Entertainment</th>
<th>Sleep</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>4</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

a. If a researcher believes the number of hours of video entertainment affects the number of hours of sleep, (that is the researcher believes the number of hours of sleep is a function of the number of hours of video entertainment), identify the independent variable and the dependent variables.

b. What are the ordered pairs for this data?

c. What are the domain and range?

d. Is this relation a function? Explain. Based on this small data set is the researcher’s assumption correct?

**Solutions**

a. Since the researcher believes the number of hours of sleep is affected by, or depends on, the number of hours of video entertainment, the independent variable is hours of video entertainment, and the dependent variable is hours of sleep.
b. The independent variable is hours of video entertainment (first coordinate) and the dependent variable is hours of sleep (second coordinate), so the set of corresponding ordered pairs is \{(8, 4), (7, 5), (2, 9), (5, 7), (4, 8), (7, 6)\}.

c. The domain is the set of first coordinates (from the independent variable) from the ordered pairs in part (b), and range is the set of second coordinates (from the dependent variable). Therefore, the domain is \{2, 4, 5, 7, 8\} and the range is \{4, 5, 6, 7, 8, 9\}.

d. This relation is not a function because one value from the domain, 7, corresponds to more than one value from the range, 5 and 6. This is shown in the ordered pairs (7, 5) and (7, 6). Based on this limited data set the researcher’s assumption is not correct, since this is not a function.

Example 7  High-Speed Internet Access

The percent of households, \(y\), with high-speed Internet access in 2007 can be modeled by the equation \(y = 0.70x + 20.03\), where \(x\) is the annual household income (in $1000s), (Source: U.S. Department of Commerce)

a. Is the relation a function? Explain.

b. If the relation is a function, identify the independent and dependent variables.

c. Use the model equation to estimate the percent of households in 2007 with high-speed Internet access (to the nearest whole percent) if the annual household income was $50,000. What point would this correspond to on the graph of the equation?

d. Determine the feasible domain.

Solutions  Video p. 8.1-15

a. The relation is a function because each input value, \(x\), can yield only one output value, \(y\).

b. The independent variable is annual household income, \(x\), the dependent variable is percent of households with high-speed Internet access (in 2007), \(y\).

c. About 55% of households with annual incomes of $50,000 in 2007 had high-speed Internet access. This corresponds to the point (50, 55) on the graph of the equation.

d. The domain is \(\{x \mid x \geq 0\}\) or \([0, \infty)\).
Section 8.1

Section 8.1 Guided Notebook

Section 8.1 Relations and Functions

Refer to the Section 8.1 printed documents that you received in class to complete the following.

Section 8.1 Objective 1: Identify Independent and Dependent Variables

When is a variable dependent?

When is a variable independent?

Example 1:
Study the solutions for example 1, and record the answers below.

For each of the following equations, identify the dependent variable and the independent variable(s).

a. \( y = 3x + 5 \)  
b. \( w = ab + 3c^2 \)  
c. \( 3x^2 + 9y = 12 \)

Section 8.1 Objective 2: Find the Domain and Range of a Relation

What is a relation?

What is the domain?

What is the range?
Carefully review Example 2.
In your own words, describe how you find the domain and range of a relation given the set of ordered pairs.

Review Example 3.
In your own words, describe how you find the domain and range of a relation given a graph.

What is a feasible domain?

**Section 8.1 Objective 3: Determine If Relations are Functions**

Define a function.

Review Example 4.
How do you determine if a set of ordered pairs is a function?

How do you determine if an equation is a function?
Section 8.1

**Section 8.1 Objective 4**: Determine If Graphs Are Functions

What is the *Vertical Line Test* and what is it used for?

Review Example 5.

(a) Draw an example of a graph that is a function and (b) a graph that is not a function.

(a) ![Graph Example A](image)

(b) ![Graph Example B](image)

**Section 8.1 Objective 5**: Solve Application Problems Involving Relations and Functions

Review Examples 6 and 7 and study their solutions.